Discussions on a special static spherically symmetric perfect fluid solution of Einstein's equations

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Abstract

In this article, a special static spherically symmetric perfect fluid solution of Einstein's equations is provided. Though pressure and density both diverge at the origin, their ratio remains constant. The solution presented here fails to give positive pressure but nevertheless, it satisfies all energy conditions. In this new spacetime geometry, the metric becomes singular at some finite value of radial coordinate although, by using isotropic coordinates, this singularity could be avoided, as has been shown here. Some characteristics of this solution are also discussed.

Key words: Einstein's gravitational field equations; Exact solution; energy conditions.

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1   Introduction

Einstein's gravitational field equations are non linear in nature. Physicists have been trying to obtain exact analytical form of the interior perfect fluid solutions for various reasons such as modelling of stars, describing astrophysical phenomena etc. Delgaty and Lake [1] have provided an excellent review of static spherically symmetric perfect fluid solutions of Einstein's equations. Among the solutions existing in literature, Schwarzschild interior solution with uniform density is the most studied for its simplicity. Nevertheless, exact solutions always demand importance for understanding of the inherent non-linear distinguishing peculiarity of gravity. In this article, we provide a discussion on a special class of exact static spherically symmetric perfect fluid solutions of Einstein's equations.

2   The basic equations and their integrals

The static spherically symmetric metric in Schwarzschild coordinates (t, r, θ, ϕ) can be written as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

Field equations pertinent to this metric are

$$e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = 8\pi\rho, \tag{2}$$

$$e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] - \frac{1}{r^2} = 8\pi p, \tag{3}$$

$$\frac{1}{2}e^{-\lambda}\left[\frac{1}{2}(\nu')^2 + \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right] = 8\pi p. \tag{4}$$

We use prime to represent derivative with respect to the radial coordinate.

The usual conservation equation $T^{ik}_{;k} = 0$ implies

$$p' = -(p + \rho)\frac{\nu'}{2} \tag{5}$$

Note that in the above set of four equations there are four unknowns, so in principle, there should exist unique exact solution. They are found to be

$$e^\nu = \frac{K}{r} \tag{6}$$

$$e^\lambda = \frac{1}{\left(\frac{C}{r} - 4\right)} \tag{7}$$

$$p = -\frac{1}{8\pi r^2} \tag{8}$$

$$\rho = \frac{5}{8\pi r^2} \tag{9}$$

where K and C are arbitrary constants.

Two plots are given below representing the dependence of metric components as functions of radial coordinate with K and C taken as parameters. Pressure and density are also plotted to show the nature of their radial dependence.
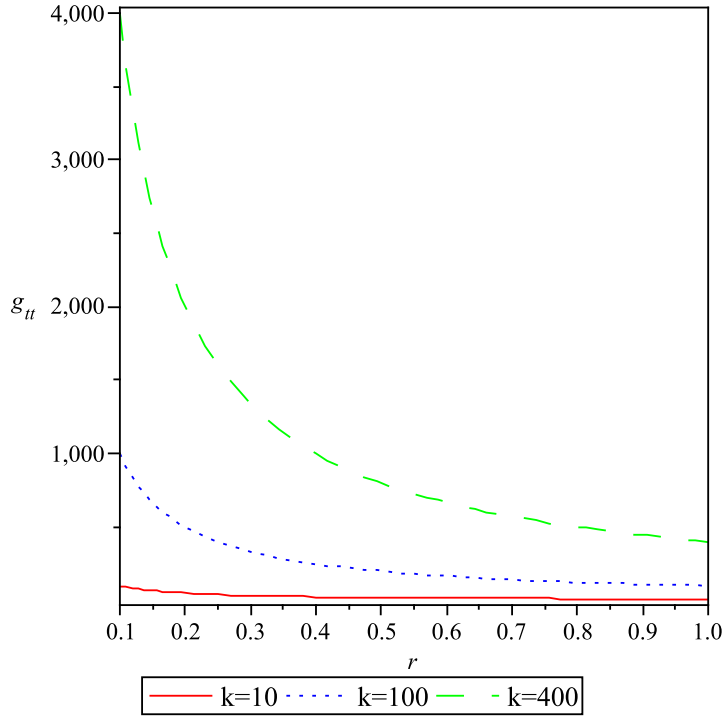

Figure 1: g_{tt} vs r for different values of K.

The pressure and density both diverge at the origin but their ratio remains constant i.e. $\frac{p}{\rho} = -\frac{1}{5}$. Here $|\, p\, |$ and ρ are monotonic decreasing functions of r. One can also see that $\rho > 0$, $p + \rho > 0$, $\rho + 3p > 0$ as well as $\rho >|\, p\, |$. Thus all energy conditions including dominant energy condition are satisfied.
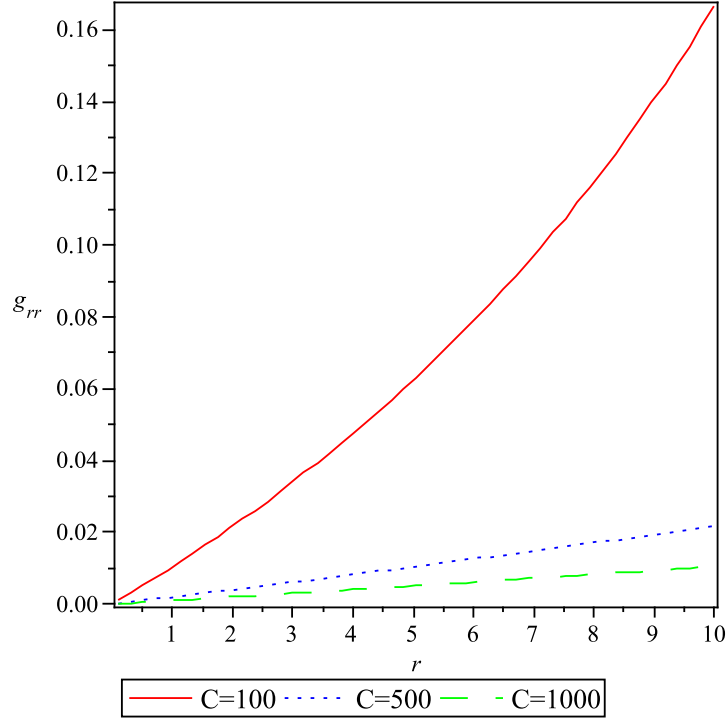
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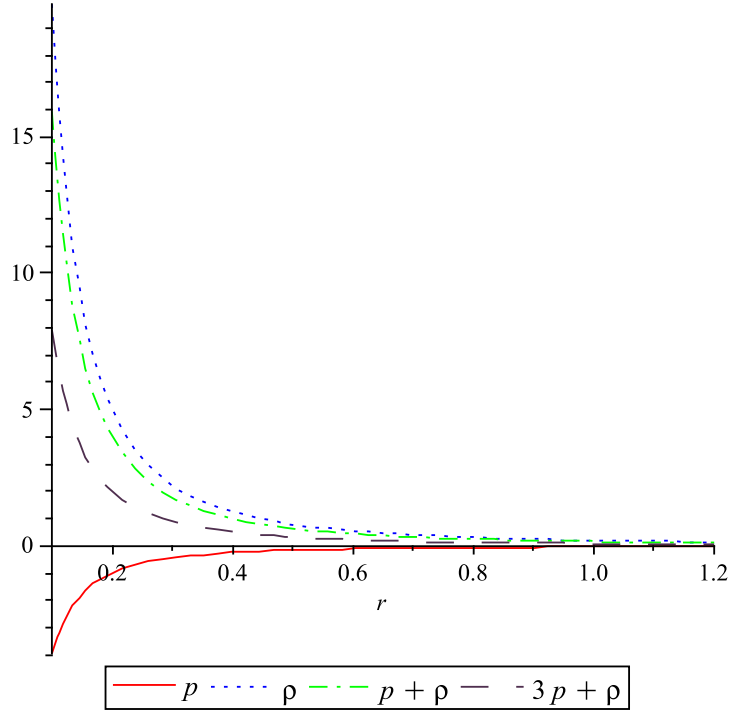Figure 2: g_{rr} vs r for different values of C.

Figure 3: p and ρ vs r

One of the important features of Einstein's equations is the appearance of curvature singularity. In this solution there is a coordinate singularity at $r = C/4$. But a close look at this spacetime solution shows that the metric does not contain any curvature singularity apart from the well known $r = 0$ singularity. (Later we shall show that in suitably chosen isotropic coordinates this singularity does not appear.) This can easily be seen by calculating the analytical expressions of the curvature invariants which, for the present metric are given by

$$R = \frac{8}{r^2}. \tag{10}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{28}{r^4}. \tag{11}$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{168}{r^4} - \frac{80C}{r^5} + \frac{12C^2}{r^6}. \tag{12}$$

The expressions show that they are regular everywhere and diverge only at $r = 0$. We also see that the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ becomes zero as $r \to \infty$. Hence the solution is asymptotically well behaved. The dependence of the scalars on the radial variable is shown in representative curve below. We avoid the $C/4$ coordinate singularity by suitably choosing the range of the radial coordiate.
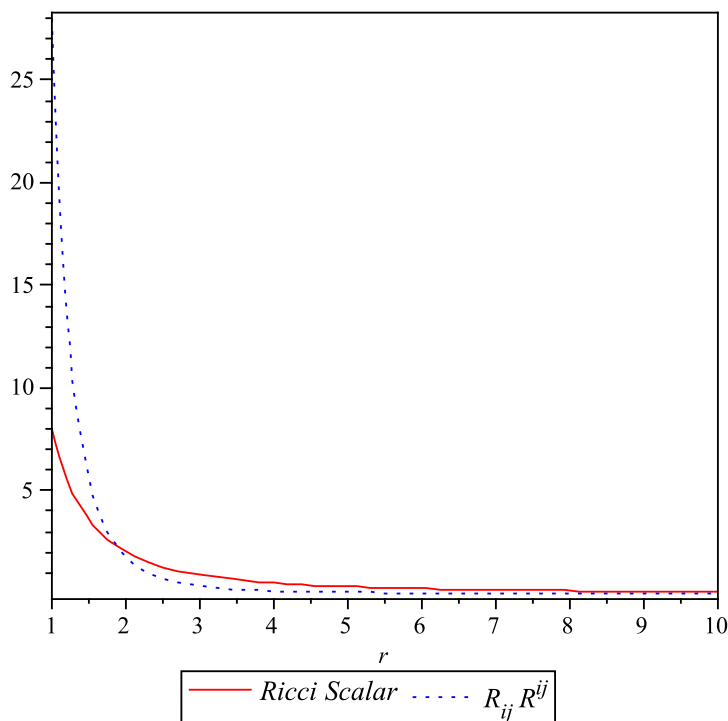

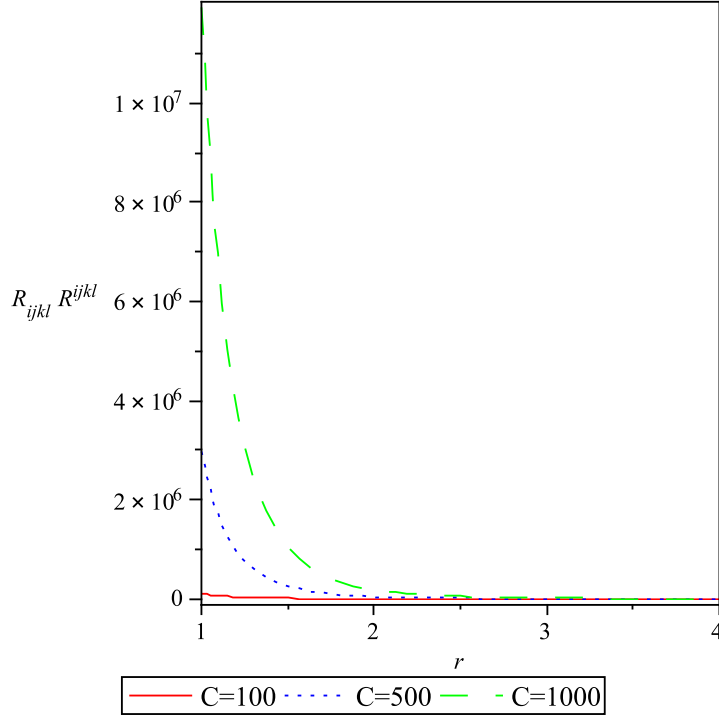Figure 4: R and $R_{\mu\nu}R^{\mu\nu}$ vs r.

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Figure 5: Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ vs r for different values of C.

3   Isotropic coordinates

The new interior metric is now written in isotropic coordinate as given by

$$ds^2 = -\left(\frac{K}{r}\right) dt^2 + [\chi(\sigma)]^2 \left[d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{13}$$

A straighforward comparison with the static spherically symmetric metric (Eq. (1)) yields

$$r^2 = \chi^2 \sigma^2, \tag{14}$$

and

$$\frac{dr^2}{\left(\frac{C}{r} - 4\right)} = \chi^2 d\sigma^2. \tag{15}$$

The above two equations yield

$$r = \frac{1}{8}\left[C + C\sin(\ln\frac{\sigma}{\sigma_0})\right], \tag{16}$$

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$$\chi^2 = \frac{\left[C + C\sin(\ln\frac{\sigma}{\sigma_0})\right]^2}{64\sigma^2}. \tag{17}$$

where σ_0 is an integration constant.

Hence, finally, the metric takes the form as

$$ds^2 = -\frac{8K\,dt^2}{\left[C + C\sin(\ln\frac{\sigma}{\sigma_0})\right]} + \frac{\left[C + C\sin(\ln\frac{\sigma}{\sigma_0})\right]^2}{64\sigma^2}\left[d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{18}$$

Note that in isotropic coordinate system, the singularity $r = \frac{C}{4}$ does not appear. This supports our earlier assertion that this singularity is a coordinate one which was evident from the calculations of the invariants.

4   Matching with exterior Schwarzschild solution

Now, we match our interior solution to the exterior Schwarzschild solution at a junction interface (S) situated at $r = a$. We impose the continuity of $g_{\mu\nu}$ across the surface S:

$g_{\mu\nu(int)}\,|_S = g_{\mu\nu(ext)}\,|_S$

at $r = a$ [i.e. on the surface S].

The continuity of the metric then gives generally

$e^{\nu}{}_{int}(a) = e^{\nu}{}_{ext}(a)$ and $e^{\lambda}{}_{int}(a) = e^{\lambda}{}_{ext}(a)$.

Hence one can find

$$\frac{K}{a} = \left(1 - \frac{2M}{a}\right) \tag{19}$$

and

$$\left(\frac{C}{a} - 4\right) = \left(1 - \frac{2M}{a}\right) \tag{20}$$

These imply,

$$K = a - 2M \tag{21}$$

$$C = 5a - 2M \tag{22}$$

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Since $r < \frac{C}{4}$, so to match our interior solution to the exterior Schwarzschild solution at a junction interface (S) situated at $r = a$, one has to take $a < \frac{C}{4}$. Equations (22) and (23) confirm this i.e. $a < \frac{C}{4}$.

Hence, our interior metric takes the form as

$$ds^2 = -\left(\frac{a - 2M}{r}\right) dt^2 + \frac{dr^2}{\left(\frac{5a-2M}{r} - 4\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (23)$$

We will face a problem to get the boundary surface (thin shell S) as $p|_{r=a} = 0$, but p never vanishes except at infinity. We see that the metric coefficients continuous at the junction i.e. at S. However, the metric need not be differentiable at the junction and the affine connection may be discontinuous there. This statement may be quantified in terms of second fundamental form of the boundary.

The second fundamental forms associated with the two sides of the shell are [2, 3]

$$K_{ij}^{\pm} = -n_\nu^{\pm} \left[\frac{\partial^2 X_\nu}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\nu} \frac{\partial X^\alpha}{\partial \xi^i} \frac{\partial X^\beta}{\partial \xi^j} \right] |_S \qquad (24)$$

where $n_\nu^{\pm}$ are the unit normals to S,

$$n_\nu^{\pm} = \pm |g^{\alpha\beta} \frac{\partial f}{\partial X^\alpha} \frac{\partial f}{\partial X^\beta}|^{-\frac{1}{2}} \frac{\partial f}{\partial X^\nu} \qquad (25)$$

with $n^\mu n_\mu = 1$.

ξ^i are the intrinsic coordinates on the shell with $f = 0$ is the parametric equation of the shell S and $-$ and $+$ corresponds to interior (our) and exterior (Schwarzschild). *Since the shell is infinitesimally thin in the radial direction there is no radial pressure.* Using Lanczos equations [2, 3], one can find the surface energy term Σ and surface tangential pressures $p_\theta = p_\phi \equiv p_t$ as

$$\Sigma = -\frac{1}{4\pi a}[\sqrt{e^{-\lambda}}]_{-}^{+},$$

$$p_t = \frac{1}{8\pi a}[(1 + \frac{a\nu'}{2})\sqrt{e^{-\lambda}}]_{-}^{+}.$$

The metric functions are continuous on S, then one finds

$$\Sigma = 0 \qquad (26)$$

and

$$p_t = \frac{1}{16\pi a} \left[\frac{2M}{a(\sqrt{1 - \frac{2M}{a}})} + \sqrt{\frac{C}{a} - 4} \right] \qquad (27)$$

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Hence one can match our interior solution with an exterior Schwarzschild solution in the presence of a thin shell. The whole spacetime is given by our metric and Schwarzschild metric which are joined smoothly.

Here the Tolman-Whittaker expression for the active gravitational mass is given by

$$M_G = \int_0^r 4 \left(T_0^0 - T_1^1 - T_2^2 - T_3^3 \right) e^{\frac{(\nu+\lambda)}{2}} r^2 dr = 2\sqrt{K} \left[\sqrt{C} - \sqrt{C - 4r} \right] \qquad (28)$$

The expression for active gravitational mass in equation(28) clearly shows that a radial dependance which is increasing function of r. We observe from the equation (28) that as $r \to 0$, $M_G \to 0$. In the interior region the maximum active gravitational mass is given by

$$M_G(Max) = 2\sqrt{K} \left[\sqrt{C} - \sqrt{C - 4a} \right] \qquad (29)$$

As $a \to \frac{C}{4}$, $M_G(Max) \to 2\sqrt{KC}$. Though the pressure and density both diverge at origin but active gravitational mass tends to zero as $r \to 0$. Thus the active gravitational mass does not suffer the well known problem of singularity.
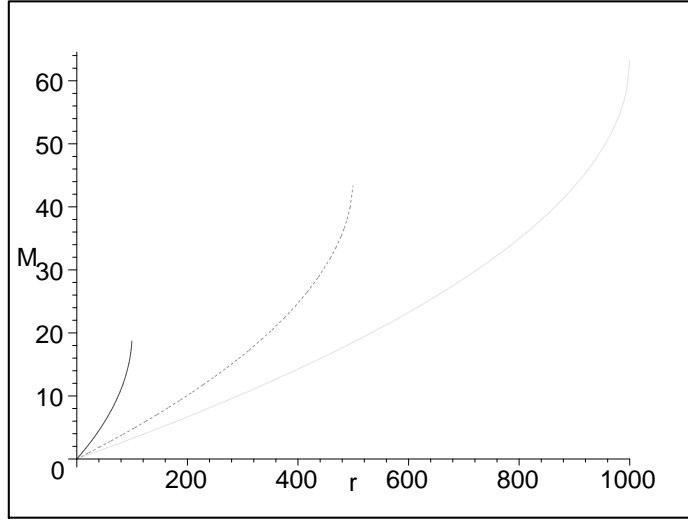

Figure 6: M_G vs r for different values of C(solid line for $C = 100$, dotted line for $C = 500$ and shaded line for $C = 1000$.

5   Concluding Remarks

We are now in position to summarize our findings.

1. One can note that at the singularity $r = \frac{C}{4}$, g_{tt} does not vanish and this implies that no horizon exists. Also all the curvature invariants are regular everywhere implying that the singularity appearing in (7) is only a coordinate singularity.

2. Our results failed to give positive pressure but all the energy conditions are satisfied for the physical acceptability of perfect fluid source. Also pressure and density failed to be regular at the origin but their ration remains constant and active gravitational mass always positive and will vanish as $r \to 0$ i.e. it does not have to tolerate the problem of singularity.

3. Though our results failed to get boundary surface where interior solution will match with exterior Schwarzschild solution, in spite of, we have shown that there exists a thin shell (boundary surface) where interior metric and Schwarzschild metric are joined smoothly.

4. One of the elementary criteria for physically acceptability spherically symmetric solution is that the subluminal sound speed to be less than unity. Here, we find that the numerical value of the subluminal sound speed , $\mid v_s^2 \mid = \mid \frac{dp}{d\rho} \mid < 1$.

The discussion above refers to the importance of the present solution in the field of physically acceptable static spherically symmetric perfect fluid solutions of Einstein equations.

Note added: After completing this work, we are informed that our solution is a special case of the solutions obtained by B Kuchowicz [4].

References

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